

HOMOLOGICAL DIMENSIONS OF MODULES OF HOLOMORPHIC FUNCTIONS ON SUBMANIFOLDS OF STEIN MANIFOLDS

A. YU. PIRKOVSKII

ABSTRACT. Let X be a Stein manifold, and let $Y \subset X$ be a closed complex submanifold. Denote by $\mathcal{O}(X)$ the algebra of holomorphic functions on X . We show that the weak (i.e., flat) homological dimension of $\mathcal{O}(Y)$ as a Fréchet $\mathcal{O}(X)$ -module equals the codimension of Y in X . In the case where X and Y are of Liouville type, the same formula is proved for the projective homological dimension of $\mathcal{O}(Y)$ over $\mathcal{O}(X)$. On the other hand, we show that if X is of Liouville type and Y is hyperconvex, then the projective homological dimension of $\mathcal{O}(Y)$ over $\mathcal{O}(X)$ equals the dimension of X .

1. INTRODUCTION

This paper is motivated by the following fact from commutative algebra. Let X be a nonsingular affine algebraic variety over \mathbb{C} , and let $\mathcal{O}(X)$ denote the algebra of regular functions on X . It is well known and easy to show that, for each nonsingular closed algebraic subvariety $Y \subset X$, the projective homological dimension of $\mathcal{O}(Y)$ considered as a module over $\mathcal{O}(X)$ is equal to the codimension of Y in X :

$$\mathrm{dh}_{\mathcal{O}(X)} \mathcal{O}(Y) = \mathrm{codim}_X Y. \quad (1)$$

Indeed, let $\mathcal{I}_Y \subset \mathcal{O}_X$ denote the ideal sheaf of Y in X . Since Y is a local complete intersection in X [17, 8.22.1], it follows that for each $y \in Y$ the ideal $\mathcal{I}_{Y,y}$ of the local ring $\mathcal{O}_{X,y}$ is generated by a regular sequence of length $m = \mathrm{codim}_X Y$. Hence we have $\mathrm{dh}_{\mathcal{O}_{X,y}} \mathcal{O}_{Y,y} = m$ [29, 3.8, Theorem 22]. The global formula (1) now follows from [28, 9.2, Theorem 11]. Note also that, since $\mathcal{O}(X)$ is Noetherian, we also have

$$\mathrm{w.dh}_{\mathcal{O}(X)} \mathcal{O}(Y) = \mathrm{codim}_X Y, \quad (2)$$

where $\mathrm{w.dh}$ stands for the weak (i.e., flat) homological dimension.

In [32], we proved (1) in the situation where X is a smooth real manifold, $\mathcal{O}(X) = C^\infty(X)$ is the algebra of smooth functions on X , and $Y \subset X$ is a closed smooth submanifold. Now the formula (1) should be understood in the context of “Topological Homology”, i.e., a relative homological algebra in categories of Fréchet modules over Fréchet algebras [18]. The above-mentioned localization technique is not applicable in this case, so we had to develop an essentially different proof which heavily relied on the softness of the structure sheaf C_X^∞ . The formula (2) also easily follows from the results of [32] (see Remark 5.14 for details).

Our goal here is to study complex analytic analogues of (1) and (2) in the context of Topological Homology. Suppose that X is a complex Stein manifold, $\mathcal{O}(X)$ is the Fréchet

2010 *Mathematics Subject Classification.* Primary 46M18; secondary 46H25, 46E25, 16E10, 16E30, 16E40.

This work was partially supported by the Ministry of Education and Science of Russia (programme “Development of the scientific potential of the Higher School”, grant no. 2.1.1/2775).

algebra of holomorphic functions on X , and $Y \subset X$ is a closed analytic submanifold. In Section 3, we show that (2) holds. As a byproduct, we obtain a complex analytic version of the Hochschild-Kostant-Rosenberg Theorem [22]. Our proof essentially uses the nuclearity of $\mathcal{O}(X)$ and some results of O. Forster [11].

Surprisingly, the validity of (1) turns out to depend on some special properties of X and Y . In Section 4, we show that (1) holds provided that both X and Y are of Liouville type (i.e., each bounded above plurisubharmonic function on X and Y is constant). In particular, this is true whenever both X and Y are affine algebraic. On the other hand, we show in Section 6 that (1) fails in the general case. Specifically, assume that X is of Liouville type and that Y is hyperconvex (this means that there exists a negative plurisubharmonic exhaustion function on Y). We show that $\mathrm{dh}_{\mathcal{O}(X)} \mathcal{O}(Y) = \dim X$ in this case. For example, if Y is the open disc embedded into \mathbb{C}^2 (cf. [1]), then $\mathrm{dh}_{\mathcal{O}(\mathbb{C}^2)} \mathcal{O}(Y) = 2$.

Our proofs heavily rely on the linear topological invariants (DN) , (Ω) , and $(\overline{\Omega})$, introduced by D. Vogt in the 1970ies (see, e.g., [25, Chapter 8]). We essentially use the splitting theorem due to D. Vogt and M. J. Wagner [41], the results of D. Vogt [44] on bounded linear maps between Fréchet spaces, and the results of V. P. Zakharyuta [47], D. Vogt [45], and A. Aytuna [2] on linear topological properties of spaces of holomorphic functions. Another essential ingredient is the Van den Bergh isomorphism [39] between the Hochschild homology and cohomology of $\mathcal{O}(X)$, which is proved in Section 5.

2. PRELIMINARIES

This section gives a brief account of some basic facts from Topological Homology. Our main reference is [18]; some details can also be found in [9, 19, 21, 33, 38].

Throughout, all vector spaces and algebras are assumed to be over the field \mathbb{C} of complex numbers. All algebras are assumed to be associative and unital. By a *Fréchet algebra* we mean an algebra A endowed with a complete, metrizable locally convex topology (i.e., A is an algebra and a Fréchet space simultaneously) such that the product map $A \times A \rightarrow A$ is continuous. If, in addition, A is locally m -convex (i.e., the topology on A can be determined by a family $\{\|\cdot\|_\lambda : \lambda \in \Lambda\}$ of seminorms satisfying $\|ab\|_\lambda \leq \|a\|_\lambda \|b\|_\lambda$ for all $a, b \in A$), then A is said to be a *Fréchet-Arens-Michael algebra*.

Let A be a Fréchet algebra. A *left Fréchet A -module* is a left A -module M endowed with a complete, metrizable locally convex topology in such a way that the action $A \times M \rightarrow M$ is continuous. We always assume that $1_A \cdot x = x$ for all $x \in M$, where 1_A is the identity of A . Left Fréchet A -modules and their continuous morphisms form a category denoted by $A\text{-mod}$. Given $M, N \in A\text{-mod}$, the space of morphisms from M to N will be denoted by $\mathbf{h}_A(M, N)$. We always endow $\mathbf{h}_A(M, N)$ with the topology of uniform convergence on bounded subsets of M ; note that this topology, in general, is not metrizable. The categories $\text{mod-}A$ and $A\text{-mod-}A$ of right Fréchet A -modules and of Fréchet A -bimodules are defined similarly. Note that $A\text{-mod-}A \cong A^e\text{-mod} \cong \text{mod-}A^e$, where $A^e = A \hat{\otimes} A^{\mathrm{op}}$, and where A^{op} stands for the algebra opposite to A .

If M is a right Fréchet A -module and N is a left Fréchet A -module, then their *A -module tensor product* $M \hat{\otimes}_A N$ is defined to be the quotient $(M \hat{\otimes} N)/L$, where $L \subset M \hat{\otimes} N$ is the closed linear span of all elements of the form $x \cdot a \otimes y - x \otimes a \cdot y$ ($x \in M$, $y \in N$, $a \in A$). As in pure algebra, the A -module tensor product can be characterized by the universal property that, for each Fréchet space E , there is a natural bijection between

the set of all continuous A -balanced bilinear maps from $M \times N$ to E and the set of all continuous linear maps from $M \widehat{\otimes}_A N$ to E .

A chain complex $C = (C_n, d_n)_{n \in \mathbb{Z}}$ in $A\text{-mod}$ is *admissible* if it splits in the category of topological vector spaces, i.e., if it has a contracting homotopy consisting of continuous linear maps. Geometrically, this means that C is exact, and $\text{Ker } d_n$ is a complemented subspace of C_n for each n .

Let \mathbf{Vect} denote the category of vector spaces and linear maps. A left Fréchet A -module P is *projective* (respectively, *strictly projective*) if the functor $\mathbf{h}_A(P, -): A\text{-mod} \rightarrow \mathbf{Vect}$ takes admissible (respectively, exact) sequences of Fréchet A -modules to exact sequences of vector spaces. Similarly, a left Fréchet A -module F is *flat* (respectively, *strictly flat*) if the tensor product functor $(-) \widehat{\otimes}_A F: \mathbf{mod}\text{-}A \rightarrow \mathbf{Vect}$ takes admissible (respectively, exact) sequences of Fréchet A -modules to exact sequences of vector spaces. Clearly, each strictly projective (respectively, strictly flat) Fréchet module is projective (respectively, flat). It is also known that every projective Fréchet module is flat.

A *resolution* of $M \in A\text{-mod}$ is a pair (P, ε) consisting of a nonnegative chain complex P in $A\text{-mod}$ and a morphism $\varepsilon: P_0 \rightarrow M$ making the sequence $P \xrightarrow{\varepsilon} M \rightarrow 0$ into an admissible complex. The *length* of P is the minimum integer n such that $P_i = 0$ for all $i > n$, or ∞ if there is no such n . If all the P_i 's are projective (respectively, flat), then (P, ε) is called a *projective resolution* (respectively, a *flat resolution*) of M . It is a standard fact that $A\text{-mod}$ has *enough projectives*, i.e., each left Fréchet A -module has a projective resolution. The same is true of $\mathbf{mod}\text{-}A$ and $A\text{-mod}\text{-}A$.

If $M, N \in A\text{-mod}$, then the space $\text{Ext}_A^n(M, N)$ is defined to be the n th cohomology of the complex $\mathbf{h}_A(P, N)$, where P is a projective resolution of M . Similarly, if $M \in \mathbf{mod}\text{-}A$ and $N \in A\text{-mod}$, then the space $\text{Tor}_n^A(M, N)$ is defined to be the n th homology of the complex $M \widehat{\otimes}_A F$, where F is a flat resolution of N . The spaces $\text{Ext}_A^n(M, N)$ and $\text{Tor}_n^A(M, N)$ do not depend on the particular choice of P and F and have the usual functorial properties (see [18] for details). If $M \in A\text{-mod}\text{-}A$, then the *n th Hochschild cohomology* (respectively, *homology*) of A with coefficients in M is defined by $\mathcal{H}^n(A, M) = \text{Ext}_{A^e}^n(A, M)$ (respectively, $\mathcal{H}_n(A, M) = \text{Tor}_n^{A^e}(M, A)$).

For each $M \in \mathbf{mod}\text{-}A$ and each $N \in A\text{-mod}$ the tensor product $N \widehat{\otimes} M$ is a Fréchet A -bimodule in a natural way, and there exist topological isomorphisms

$$\text{Tor}_n^A(M, N) \cong \mathcal{H}_n(A, N \widehat{\otimes} M). \quad (3)$$

If $M, N \in A\text{-mod}$ and M is a Banach module, then $\mathcal{L}(M, N)$ is a Fréchet A -bimodule in a natural way, and we have vector space isomorphisms

$$\text{Ext}_A^n(M, N) \cong \mathcal{H}^n(A, \mathcal{L}(M, N)). \quad (4)$$

In some cases, the spaces $\text{Tor}_n^A(M, N)$ can be computed via nonadmissible resolutions. Let $N \in A\text{-mod}$, and let (F, ε) be a pair consisting of a nonnegative chain complex F in $A\text{-mod}$ and a morphism $\varepsilon: F_0 \rightarrow N$ making the sequence $F \xrightarrow{\varepsilon} N \rightarrow 0$ into an exact complex. Suppose also that all the F_i 's are flat Fréchet modules (so that F is “almost” a flat resolution of N). Take $M \in \mathbf{mod}\text{-}A$, and assume that either all the F_i 's are nuclear, or both A and M are nuclear. Then we have $\text{Tor}_n^A(M, N) \cong H_n(M \widehat{\otimes}_A F)$ (see [38] or [9, 3.1.13]).

The *projective homological dimension* of $M \in A\text{-mod}$ is the minimum integer $n = \text{dh}_A M \in \mathbb{Z}_+ \cup \{\infty\}$ with the property that M has a projective resolution of length n . Similarly, the *weak homological dimension* of $M \in A\text{-mod}$ is the minimum integer

$n = \text{w.dh}_A M \in \mathbb{Z}_+ \cup \{\infty\}$ with the property that M has a flat resolution of length n . Equivalently,

$$\begin{aligned} \text{dh}_A M &= \min\{n \in \mathbb{Z}_+ \mid \text{Ext}_A^{n+1}(M, N) = 0 \ \forall N \in A\text{-mod}\} \\ &= \min\{n \in \mathbb{Z}_+ \mid \text{Ext}_A^{p+1}(M, N) = 0 \ \forall N \in A\text{-mod}, \ \forall p \geq n\}; \\ \text{w.dh}_A M &= \min \left\{ n \in \mathbb{Z}_+ \mid \begin{array}{l} \text{Tor}_{n+1}^A(N, M) = 0, \text{ and } \text{Tor}_n^A(N, M) \\ \text{is Hausdorff } \forall N \in \text{mod-}A \end{array} \right\} \\ &= \min \left\{ n \in \mathbb{Z}_+ \mid \begin{array}{l} \text{Tor}_{p+1}^A(N, M) = 0, \text{ and } \text{Tor}_n^A(N, M) \\ \text{is Hausdorff } \forall N \in \text{mod-}A, \ \forall p \geq n \end{array} \right\}. \end{aligned}$$

Note that $\text{dh}_A M = 0$ if and only if M is projective, and $\text{w.dh}_A M = 0$ if and only if M is flat. Since each projective module is flat, we clearly have $\text{w.dh}_A M \leq \text{dh}_A M$.

The *global dimension* and the *weak global dimension* of A are defined by

$$\begin{aligned} \text{dg } A &= \sup\{\text{dh}_A M \mid M \in A\text{-mod}\}, \\ \text{w.dg } A &= \sup\{\text{w.dh}_A M \mid M \in A\text{-mod}\}. \end{aligned}$$

The *bidimension* and the *weak bidimension* of A are defined by $\text{db } A = \text{dh}_{A^e} A$ and $\text{w.db } A = \text{w.dh}_{A^e} A$, respectively. We clearly have $\text{w.dg } A \leq \text{dg } A$ and $\text{w.db } A \leq \text{db } A$. It is also true (but less obvious) that $\text{dg } A \leq \text{db } A$ and $\text{w.dg } A \leq \text{w.db } A$.

Throughout the paper, all complex manifolds are assumed to be connected (although most of our results can easily be extended to manifolds having finitely many components of the same dimension). The structure sheaf of a complex manifold X will always be denoted by \mathcal{O}_X . The phrase “an \mathcal{O}_X -module” will mean “a sheaf of \mathcal{O}_X -modules”. Recall that $\mathcal{O}(X)$ is a nuclear Fréchet algebra with respect to the compact-open topology (i.e., the topology of uniform convergence on compact subsets of X). Recall also (see, e.g., [13, 5.6]) that, for each coherent \mathcal{O}_X -module \mathcal{F} , the space of global sections $\mathcal{F}(X) = \Gamma(X, \mathcal{F})$ has a canonical topology making $\mathcal{F}(X)$ into a nuclear Fréchet $\mathcal{O}(X)$ -module.

If X is a Stein manifold, then a Fréchet $\mathcal{O}(X)$ -module F is a *Stein module* if it is topologically isomorphic to $\mathcal{F}(X)$ for some coherent \mathcal{O}_X -module \mathcal{F} . By a result of O. Forster [11, 2.1], the functor $\Gamma(X, \cdot)$ of global sections is an equivalence between the category of coherent \mathcal{O}_X -modules and the full subcategory of $\mathcal{O}(X)\text{-mod}$ consisting of Stein $\mathcal{O}(X)$ -modules.

3. FORSTER RESOLUTION AND WEAK DIMENSION

The following lemma is an easy consequence of O. Forster’s results [11]. We formulate it here for the reader’s convenience.

Lemma 3.1. *Let X be a Stein manifold, and let $Y \subset X$ be a closed submanifold of codimension m . There exists an exact sequence*

$$0 \leftarrow \mathcal{O}_Y \leftarrow \mathcal{P}_0 \leftarrow \mathcal{P}_1 \leftarrow \cdots \leftarrow \mathcal{P}_m \leftarrow 0 \quad (5)$$

where $\mathcal{P}_0, \dots, \mathcal{P}_m$ are locally free \mathcal{O}_X -modules.

Proof. By [11, 6.4], there exists an exact sequence

$$0 \leftarrow \mathcal{O}_Y \leftarrow \mathcal{F}_0 \xleftarrow{d_0} \mathcal{F}_1 \leftarrow \cdots \leftarrow \mathcal{F}_{m-1} \xleftarrow{d_{m-1}} \mathcal{F}_m \xleftarrow{d_m} \cdots$$

where $\mathcal{F}_0, \mathcal{F}_1, \dots$ are free \mathcal{O}_X -modules. Letting $\mathcal{P}_m = \mathcal{I}m d_{m-1}$, we obtain an exact sequence

$$0 \leftarrow \mathcal{O}_Y \leftarrow \mathcal{F}_0 \leftarrow \mathcal{F}_1 \leftarrow \dots \leftarrow \mathcal{F}_{m-1} \leftarrow \mathcal{P}_m \leftarrow 0. \quad (6)$$

To complete the proof, it remains to show that \mathcal{P}_m is locally free. Fix $x \in X$, and consider the exact sequence

$$0 \leftarrow \mathcal{O}_{Y,x} \leftarrow \mathcal{F}_{0,x} \leftarrow \mathcal{F}_{1,x} \leftarrow \dots \leftarrow \mathcal{F}_{m-1,x} \leftarrow \mathcal{P}_{m,x} \leftarrow 0 \quad (7)$$

of $\mathcal{O}_{X,x}$ -modules. If $x \notin Y$, then $\mathcal{O}_{Y,x} = 0$, whence (7) splits, and so $\mathcal{P}_{m,x}$ is projective. Now suppose that $x \in Y$, and let $\mathcal{I} \subset \mathcal{O}_X$ denote the ideal sheaf of Y . We have $\mathcal{O}_{Y,x} \cong \mathcal{O}_{X,x}/\mathcal{I}_x$. Choose a local coordinate system z^1, \dots, z^n in a neighborhood U of x such that $Y \cap U = \{z \in U : z^1 = \dots = z^m = 0\}$. The $\mathcal{O}_{X,x}$ -module \mathcal{I}_x is generated by the regular sequence (z^1, \dots, z^m) , which implies that $\text{dh}_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,x} = m$ (see, e.g., [29, 3.8, Theorem 22])¹. By looking at (7), we conclude that $\mathcal{P}_{m,x}$ is projective.

Thus for each $x \in X$ the $\mathcal{O}_{X,x}$ -module $\mathcal{P}_{m,x}$ is projective. Since $\mathcal{O}_{X,x}$ is local, this means that $\mathcal{P}_{m,x}$ is free. Therefore \mathcal{P}_m is a locally free \mathcal{O}_X -module, as required. \square

Corollary 3.2. *Let X be a Stein manifold, and let $Y \subset X$ be a closed submanifold of codimension m . There exists an exact sequence*

$$0 \leftarrow \mathcal{O}(Y) \leftarrow P_0 \leftarrow P_1 \leftarrow \dots \leftarrow P_m \leftarrow 0 \quad (8)$$

of Fréchet $\mathcal{O}(X)$ -modules, where P_0, \dots, P_m are finitely generated and strictly projective.

Proof. Applying the global section functor $\Gamma(X, \cdot)$ to (5), we obtain (8). Using [11, 6.2 and 6.3], we conclude that P_0, \dots, P_m are finitely generated and strictly projective. \square

Definition 3.3. Any sequence of the form (5) (respectively, (8)) will be called a *Forster resolution* of \mathcal{O}_Y over \mathcal{O}_X (respectively, of $\mathcal{O}(Y)$ over $\mathcal{O}(X)$).

Remark 3.4. Since the global section functor is an equivalence between the category of coherent \mathcal{O}_X -modules and the category of Stein $\mathcal{O}(X)$ -modules, it yields a 1-1 correspondence between Forster resolutions of \mathcal{O}_Y over \mathcal{O}_X and Forster resolutions of $\mathcal{O}(Y)$ over $\mathcal{O}(X)$.

Remark 3.5. In fact, it easily follows from Forster's construction [11] that resolution (8) can be chosen in such a way that $P_0 = \mathcal{O}(X)$ and that the arrow $P_0 \rightarrow \mathcal{O}(Y)$ is the restriction map. We will not use this in the sequel.

Remark 3.6. Note that (8) is not necessarily a resolution in the sense of Topological Homology, i.e., it need not be admissible. The first counterexample was given in [27, Proposition 5.3]; a more general situation will be discussed in Remark 6.4 below.

Proposition 3.7. *Let X be a Stein manifold, and let Y be a closed submanifold of X . Then for each coherent \mathcal{O}_X -module \mathcal{F} and each $p \in \mathbb{Z}_+$ we have a topological isomorphism*

$$\text{Tor}_p^{\mathcal{O}(X)}(\mathcal{O}(Y), \mathcal{F}(X)) \cong \Gamma(X, \mathcal{T}or_p^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{F})).$$

¹Here, of course, the projective homological dimension $\text{dh}_{\mathcal{O}_{X,x}} \mathcal{O}_{Y,x}$ should be understood in the purely algebraic context.

Proof. Let $\mathcal{P} \rightarrow \mathcal{O}_Y \rightarrow 0$ be a Forster resolution of \mathcal{O}_Y over \mathcal{O}_X . Applying the global section functor, we get a Forster resolution $\mathcal{P}(X) \rightarrow \mathcal{O}(Y) \rightarrow 0$ of $\mathcal{O}(Y)$ over $\mathcal{O}(X)$ (see Remark 3.4). Since all the modules $\mathcal{P}_i(X)$ are nuclear, it follows that

$$\mathrm{Tor}_p^{\mathcal{O}(X)}(\mathcal{O}(Y), \mathcal{F}(X)) \cong H_p(\mathcal{P}(X) \hat{\otimes}_{\mathcal{O}(X)} \mathcal{F}(X)).$$

We also have $\mathcal{P}(X) \hat{\otimes}_{\mathcal{O}(X)} \mathcal{F}(X) \cong \Gamma(X, \mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{F})$ (see, e.g., [9, 4.2.4] or [31, 2.2]). Hence

$$\begin{aligned} \mathrm{Tor}_p^{\mathcal{O}(X)}(\mathcal{O}(Y), \mathcal{F}(X)) &\cong H_p(\Gamma(X, \mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{F})) \\ &\cong \Gamma(X, H_p(\mathcal{P} \otimes_{\mathcal{O}_X} \mathcal{F})) = \Gamma(X, \mathcal{T}or_p^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{F})). \end{aligned} \quad \square$$

Corollary 3.8. *Let X be a Stein manifold, and let Y be a closed submanifold of X . Denote by $\mathcal{I} \subset \mathcal{O}_X$ the ideal sheaf of Y . Then for each $p \in \mathbb{Z}_+$ we have*

$$\mathrm{Tor}_p^{\mathcal{O}(X)}(\mathcal{O}(Y), \mathcal{O}(Y)) \cong \Gamma(X, \bigwedge^p(\mathcal{I}/\mathcal{I}^2)).$$

Proof. By [24, 3.5], we have $\mathcal{T}or_p^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{O}_Y) \cong \bigwedge^p(\mathcal{I}/\mathcal{I}^2)$ (the proof of this fact, given in [24] for regular schemes, applies to complex manifolds without changes). Now it remains to apply Proposition 3.7. \square

As a byproduct, we obtain the following analytic version of the Hochschild–Kostant–Rosenberg Theorem [22].

Corollary 3.9. *Let X be a Stein manifold. Then for each $p \in \mathbb{Z}_+$ we have*

$$\mathcal{H}_p(\mathcal{O}(X), \mathcal{O}(X)) \cong \Omega^p(X),$$

where $\Omega^p(X)$ is the space of holomorphic p -forms on X .

Proof. Recall from [14, II.3.3] that there exists a topological isomorphism

$$\mathcal{O}(X)^e = \mathcal{O}(X) \hat{\otimes} \mathcal{O}(X) \cong \mathcal{O}(X \times X), \quad f \otimes g \mapsto ((x, y) \mapsto f(x)g(y)).$$

Under this identification, the $\mathcal{O}(X)^e$ -module $\mathcal{O}(X)$ becomes the $\mathcal{O}(X \times X)$ -module $\mathcal{O}(\Delta)$, where $\Delta = \{(x, x) : x \in X\}$ is the diagonal of $X \times X$. Let $\mathcal{I} \subset \mathcal{O}_{X \times X}$ be the ideal sheaf of Δ . Identifying X with Δ via the map $x \mapsto (x, x)$ induces a sheaf isomorphism between $(\mathcal{I}/\mathcal{I}^2)|_\Delta$ and the cotangent sheaf Ω_X^1 (see, e.g., [15]). Now Corollary 3.8 implies that

$$\begin{aligned} \mathcal{H}_p(\mathcal{O}(X), \mathcal{O}(X)) &= \mathrm{Tor}_p^{\mathcal{O}(X)^e}(\mathcal{O}(X), \mathcal{O}(X)) \cong \mathrm{Tor}_p^{\mathcal{O}(X \times X)}(\mathcal{O}(\Delta), \mathcal{O}(\Delta)) \\ &\cong \Gamma(X \times X, \bigwedge^p(\mathcal{I}/\mathcal{I}^2)) \cong \Omega^p(X). \end{aligned} \quad \square$$

Theorem 3.10. *Let X be a Stein manifold, and let Y be a closed submanifold of X . Then $\mathrm{w.dh}_{\mathcal{O}(X)} \mathcal{O}(Y) = \mathrm{codim}_X Y$.*

Proof. Let M be a Fréchet $\mathcal{O}(X)$ -module, and let $P \rightarrow \mathcal{O}(Y) \rightarrow 0$ be a Forster resolution of $\mathcal{O}(Y)$ over $\mathcal{O}(X)$. Using the nuclearity argument (see the proof of Proposition 3.7), we see that $\mathrm{Tor}_p^{\mathcal{O}(X)}(\mathcal{O}(Y), M)$ is topologically isomorphic to $H_p(P \hat{\otimes}_{\mathcal{O}(X)} M)$, which vanishes for $p > m$ and is Hausdorff for $p = m$, where $m = \mathrm{codim}_X Y$. Therefore $\mathrm{w.dh}_{\mathcal{O}(X)} \mathcal{O}(Y) \leq m$. To obtain the opposite estimate, recall from [24, 3.4] that $(\mathcal{I}/\mathcal{I}^2)|_Y$ is locally free of rank m . In particular, $\bigwedge^m(\mathcal{I}/\mathcal{I}^2) \neq 0$, and it follows from Corollary 3.8 that $\mathrm{Tor}_m^{\mathcal{O}(X)}(\mathcal{O}(Y), \mathcal{O}(Y)) \neq 0$. The rest is clear. \square

Corollary 3.11 ([31, 36]). *Let X be a Stein manifold. Then $\text{w.dg } \mathcal{O}(X) = \text{w.db } \mathcal{O}(X) = \dim X$.*

Proof. As in the proof of Corollary 3.9, identify X with the diagonal $\Delta \subset X \times X$. Theorem 3.10 implies that

$$\text{w.db } \mathcal{O}(X) = \text{w.dh}_{\mathcal{O}(X)^e} \mathcal{O}(X) = \text{w.dh}_{\mathcal{O}(X \times X)} \mathcal{O}(\Delta) = \text{codim}_{X \times X} \Delta = \dim X.$$

Hence $\text{w.dg } \mathcal{O}(X) \leq \dim X$. Applying Theorem 3.10 to the singleton $Y = \{x_0\}$ yields the opposite estimate $\text{w.dg } \mathcal{O}(X) \geq \text{codim}_X \{x_0\} = \dim X$. \square

4. LIOUVILLE-TYPE PROPERTY AND PROJECTIVE DIMENSION

Let X be a complex manifold. Following [30], we say that X is of *Liouville type* if each bounded above plurisubharmonic function on X is constant. For example, each nonsingular affine algebraic variety is of Liouville type. More examples can be found in [3, 4] (note that Stein manifolds of Liouville type are called *parabolic* in [4]).

Starting from this section, we will make use of the linear topological invariants (DN) and (Ω) introduced by D. Vogt [40, 41] (see also [25]). Let E be a Fréchet space. By definition, E has property (DN) if the topology on E can be determined by an increasing sequence $\{\|\cdot\|_n : n \in \mathbb{N}\}$ of seminorms satisfying the following condition: there exists $p \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ and each $0 < r < 1$ there exist $n \in \mathbb{N}$ and $C > 0$ such that

$$\|x\|_k \leq C \|x\|_p^r \|x\|_n^{1-r} \quad (x \in E).$$

Given a continuous seminorm $\|\cdot\|$ on a Fréchet space E , define the dual “seminorm” $\|\cdot\|^*: E^* \rightarrow [0, +\infty]$ by $\|y\|^* = \sup\{|y(x)| : \|x\| \leq 1\}$. Note that $\|\cdot\|^*$ can take the value $+\infty$ as well. By definition, E has property (Ω) if the topology on E can be determined by an increasing sequence $\{\|\cdot\|_n : n \in \mathbb{N}\}$ of seminorms satisfying the following condition: for each $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ there exist $C > 0$ and $r \in (0, 1)$ satisfying

$$\|y\|_q^* \leq C (\|y\|_p^*)^r (\|y\|_k^*)^{1-r} \quad (y \in E^*).$$

Property (DN) is inherited by subspaces, while (Ω) is inherited by quotients. A basic fact about (DN) and (Ω) is the following Splitting Theorem due to Vogt and Wagner [41] (see also [25]): an exact sequence $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$ of nuclear Fréchet spaces splits provided that E has (Ω) and G has (DN) .

The following lemma is an easy consequence of the Splitting Theorem.

Lemma 4.1. *Let $E = (E_i, d_i)$ be a chain complex of nuclear Fréchet spaces. Suppose that all the spaces E_i have properties (DN) and (Ω) . Then E splits.*

Proof. For each $i \in \mathbb{Z}$ we have an exact sequence

$$0 \leftarrow K_i \xleftarrow{d_i} E_{i+1} \leftarrow K_{i+1} \leftarrow 0, \quad (9)$$

where $K_i = \text{Im}(d_i: E_{i+1} \rightarrow E_i)$. Note that each K_i , being a subspace of E_i and a quotient of E_{i+1} , has properties (DN) and (Ω) . Now the Splitting Theorem implies that (9) splits for all $i \in \mathbb{Z}$, which means exactly that E splits. \square

Let now X be a Stein manifold. As was observed in [42], $\mathcal{O}(X)$ always has property (Ω) . Indeed, for $X = \mathbb{C}^n$ this can be seen directly, and in the general case X can be embedded into \mathbb{C}^n for sufficiently large n , so $\mathcal{O}(X)$ becomes a quotient of $\mathcal{O}(\mathbb{C}^n)$. On the other hand, it was shown independently by Zakharyuta [47], Vogt [45], and Aytuna [2]

that $\mathcal{O}(X)$ has (DN) if and only if X is of Liouville type. In particular, this is true provided that X is affine algebraic [46]. For more results in this direction, see [3, 4, 30].

Corollary 4.2. *Let X be a Stein manifold, and let Y be a closed submanifold of X . Suppose that both X and Y are of Liouville type. Then each Forster resolution of $\mathcal{O}(Y)$ over $\mathcal{O}(X)$ is admissible.*

Proof. Since X and Y are of Liouville type, it follows that $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ have properties (DN) and (Ω) . Each finitely generated, strictly projective Fréchet $\mathcal{O}(X)$ -module also has properties (DN) and (Ω) , because it is isomorphic to a direct summand of $\mathcal{O}(X)^p$ for some $p \in \mathbb{N}$. Now we see that resolution (8) satisfies the conditions of Lemma 4.1. \square

Theorem 4.3. *Let X be a Stein manifold, and let Y be a closed submanifold of X . Suppose that both X and Y are of Liouville type. Then $\mathrm{dh}_{\mathcal{O}(X)} \mathcal{O}(Y) = \mathrm{codim}_X Y$.*

Proof. Corollary 4.2 implies that $\mathrm{dh}_{\mathcal{O}(X)} \mathcal{O}(Y) \leq \mathrm{codim}_X Y$, and the opposite estimate is immediate from Theorem 3.10. \square

Arguing in the same way as in the proof of Corollary 3.11, we obtain the following.

Corollary 4.4. *Let X be a Stein manifold of Liouville type. Then $\mathrm{dg} \mathcal{O}(X) = \mathrm{db} \mathcal{O}(X) = \dim X$.*

5. VAN DEN BERGH ISOMORPHISMS FOR $\mathcal{O}(X)$

The Van den Bergh isomorphisms are certain relations between Hochschild homology and cohomology of associative algebras. Special cases of such isomorphisms can be traced back to the origins of homological algebra, but systematically they were studied only in 1998 by M. Van den Bergh [39]. For more recent results involving the Van den Bergh isomorphisms, see [5, 7, 8, 10, 12, 23, 26], to cite a few. Below we will use a Fréchet algebra version of the Van den Bergh isomorphisms, which was introduced and applied in [32] to some problems of Topological Homology (see also [35]).

Let A be a Fréchet algebra. A Fréchet A -bimodule M is *invertible* if there exists a Fréchet A -bimodule M^{-1} such that $M \hat{\otimes}_A M^{-1} \cong M^{-1} \hat{\otimes}_A M \cong A$ in $A\text{-mod-}A$. For example, if X is a Stein manifold, then for each line sheaf \mathcal{L} of \mathcal{O}_X -modules the $\mathcal{O}(X)$ -bimodule $\mathcal{L}(X)$ is invertible, and $\mathcal{L}(X)^{-1} = \mathcal{L}^{-1}(X)$, where $\mathcal{L}^{-1} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$. More examples can be found in [35]. By [32, Prop. 5.2.6], each invertible Fréchet A -bimodule is projective in $A\text{-mod}$ and in $\text{mod-}A$.

Following [10] (see also [35] for the Fréchet algebra case), we say that A *satisfies the Van den Bergh condition* $\mathrm{VdB}(n)$ if there exists an invertible Fréchet A -bimodule L (a *dualizing bimodule*) such that for each Fréchet A -bimodule M and each $i \in \mathbb{Z}$ there is a vector space isomorphism

$$\mathcal{H}^i(A, M) \cong \mathcal{H}_{n-i}(A, L \hat{\otimes}_A M).$$

For example, if X is a smooth manifold, then $C^\infty(X)$ satisfies $\mathrm{VdB}(n)$ with $n = \dim X$ and $L = T^n(X)$, the module of smooth n -polyvector fields on X [32]. We refer to [35] for more examples.

The goal of this section is to establish the Van den Bergh isomorphisms for $A = \mathcal{O}(X)$, where X is a Stein manifold of Liouville type.

Lemma 5.1. *Let A be a Fréchet algebra, and let $C = (C^i, d^i)$ be a cochain complex in $\mathbf{mod}\text{-}A$ such that the cohomology spaces $H^i(C)$ are Hausdorff. Let $F \in A\text{-mod}$ be a flat Fréchet module, and assume that either all the C^i 's are nuclear or A and F are nuclear. Then the cohomology spaces $H^i(C \widehat{\otimes}_A F)$ are also Hausdorff, and there exist canonical topological isomorphisms*

$$H^i(C) \widehat{\otimes}_A F \cong H^i(C \widehat{\otimes}_A F). \quad (10)$$

Specifically, if $\xi \in H^i(C)$ is represented by an i -cocycle $c \in C^i$, then (10) takes $\xi \otimes_A x$ to the cohomology class of $c \otimes_A x$.

Proof. Let $Z^i = Z^i(C)$ denote the space of i -cocycles of C . Write also $H^i = H^i(C)$ for short. For each i we have an exact sequence

$$0 \rightarrow Z^i \rightarrow C^i \xrightarrow{d^i} Z^{i+1} \rightarrow H^{i+1} \rightarrow 0$$

of right Fréchet A -modules. Since F is flat, it follows from the nuclearity assumptions (see, e.g., [9, 3.1.12]) that the tensored sequence

$$0 \rightarrow Z^i \widehat{\otimes}_A F \rightarrow C^i \widehat{\otimes}_A F \xrightarrow{d^i \otimes_A 1_F} Z^{i+1} \widehat{\otimes}_A F \rightarrow H^{i+1} \widehat{\otimes}_A F \rightarrow 0 \quad (11)$$

is exact. Using the exactness of (11) first for i , and then for $i+1$, we obtain topological isomorphisms

$$\begin{aligned} Z^i \widehat{\otimes}_A F &\cong \text{Ker}(C^i \widehat{\otimes}_A F \xrightarrow{d^i \otimes_A 1_F} Z^{i+1} \widehat{\otimes}_A F) \\ &= \text{Ker}(C^i \widehat{\otimes}_A F \xrightarrow{d^i \otimes_A 1_F} C^{i+1} \widehat{\otimes}_A F) = Z^i(C \widehat{\otimes}_A F). \end{aligned} \quad (12)$$

Now it follows from the exactness of (11) for $i-1$ and from (12) that

$$\begin{aligned} H^i \widehat{\otimes}_A F &\cong \text{Coker}(C^{i-1} \widehat{\otimes}_A F \xrightarrow{d^{i-1} \otimes_A 1_F} Z^i \widehat{\otimes}_A F) \\ &\cong \text{Coker}(C^{i-1} \widehat{\otimes}_A F \xrightarrow{d^{i-1} \otimes_A 1_F} Z^i(C \widehat{\otimes}_A F)) = H^i(C \widehat{\otimes}_A F). \end{aligned}$$

It follows from the construction that, for each $c \in Z^i$ and $x \in F$, the resulting isomorphism (10) indeed takes $(c + \text{Im } d^{i-1}) \otimes_A x$ to $c \otimes_A x + \text{Im}(d^{i-1} \otimes_A 1_F)$, as required. \square

In what follows, we will need some topological modules which are not Fréchet modules. Let A be a Fréchet algebra. A *left A - $\widehat{\otimes}$ -module* is a left A -module M endowed with a complete locally convex topology in such a way that the action $A \times M \rightarrow M$ is continuous. Right A - $\widehat{\otimes}$ -modules and A - $\widehat{\otimes}$ -bimodules are defined similarly. If M is a right A - $\widehat{\otimes}$ -module and N is a left A - $\widehat{\otimes}$ -module, then their A -module tensor product $M \widehat{\otimes}_A N$ is defined similarly to the case of Fréchet modules (see Section 2). The only exception is that the quotient $(M \widehat{\otimes} N)/L$ need not be complete in the nonmetrizable case, so $M \widehat{\otimes}_A N$ is defined to be the completion of $(M \widehat{\otimes} N)/L$.

Recall now a construction from [32]. Let A be a Fréchet algebra, and let $M, N \in A\text{-mod}$. Fix a projective resolution $P \rightarrow M \rightarrow 0$ of M in $A\text{-mod}$. Each $\mathbf{h}_A(P_i, A)$ has a natural structure of a right A - $\widehat{\otimes}$ -module given by $(\varphi \cdot a)(p) = \varphi(p)a$ for $a \in A, p \in P$. We have a map of complexes

$$\mathbf{h}_A(P, A) \widehat{\otimes}_A N \rightarrow \mathbf{h}_A(P, N), \quad \varphi \otimes_A n \mapsto (p \mapsto \varphi(p) \cdot n). \quad (13)$$

If, for some n , the spaces $\text{Ext}_A^n(M, A)$ and $\text{Ext}_A^n(M, N)$ are Hausdorff and complete, then $\text{Ext}_A^n(M, A)$ becomes a right A - $\widehat{\otimes}$ -module in a natural way, and we have continuous linear maps

$$\text{Ext}_A^n(M, A) \widehat{\otimes}_A N \rightarrow H^n(\mathbf{h}_A(P, A) \widehat{\otimes}_A N) \rightarrow \text{Ext}_A^n(M, N). \quad (14)$$

Here, for each $\xi \in \text{Ext}_A^n(M, A)$ represented by an n -cocycle $\varphi \in \mathbf{h}_A(P_n, A)$, the first arrow in (14) takes $\xi \otimes_A n$ to the cohomology class of $\varphi \otimes_A n$. The second arrow in (14) is induced by (13). The composite map

$$\text{Ext}_A^n(M, A) \widehat{\otimes}_A N \rightarrow \text{Ext}_A^n(M, N) \quad (15)$$

does not depend on the choice of the projective resolution P .

Lemma 5.2. *Let A be a Fréchet algebra, and let P be a finitely generated, strictly projective left Fréchet A -module. Then for each $N \in A\text{-mod}$ $\mathbf{h}_A(P, N)$ is a Fréchet space. Moreover, if N is nuclear, then so is $\mathbf{h}_A(P, N)$.*

Proof. If $P = A$, then $\mathbf{h}_A(A, N)$ is topologically isomorphic to N via the map $\varphi \mapsto \varphi(1)$; the continuity of the inverse map $n \mapsto (a \mapsto a \cdot n)$ is easily checked. The general case follows by additivity and functoriality. \square

Following [35], we say that a left Fréchet A -module M is of *strictly finite type* if M has a resolution $P \rightarrow M \rightarrow 0$ consisting of finitely generated, strictly projective Fréchet A -modules. The Fréchet algebra A is said to be of *finite type* if A is of strictly finite type in $A^e\text{-mod}$. For example, if X is a Stein manifold, $Y \subset X$ is a closed submanifold, and both X and Y are of Liouville type, then $\mathcal{O}(Y)$ is of strictly finite type over $\mathcal{O}(X)$ (see Corollary 4.2). Identifying $\mathcal{O}(X)^e$ with $\mathcal{O}(X \times X)$, we see, in particular, that the algebra $\mathcal{O}(X)$ is of finite type.

Corollary 5.3. *Let A be a Fréchet algebra, and let $M \in A\text{-mod}$ be of strictly finite type. If $\text{Ext}_A^n(M, N)$ is Hausdorff for some n , then $\text{Ext}_A^n(M, N)$ is a Fréchet space.*

Lemma 5.4. *Let A be a nuclear Fréchet algebra, and let $M \in A\text{-mod}$ be of strictly finite type. Suppose that $\text{Ext}_A^n(M, A)$ is Hausdorff for all $n \in \mathbb{Z}_+$. Then for each flat $N \in A\text{-mod}$ and each $n \in \mathbb{Z}_+$ $\text{Ext}_A^n(M, N)$ is Hausdorff as well, and the canonical map (15) is a topological isomorphism.*

Proof. Fix a resolution $P \rightarrow M \rightarrow 0$ consisting of finitely generated, strictly projective Fréchet A -modules. By Lemma 5.2, the cochain complex $C = \mathbf{h}_A(P, A)$ satisfies the conditions of Lemma 5.1. Therefore the first arrow in (14) is a topological isomorphism. Since each P_i is finitely generated and strictly projective, it follows that (13) is also a topological isomorphism (cf. the proof of Lemma 5.2). Hence the second arrow in (14) is a topological isomorphism as well. This completes the proof. \square

For the reader's convenience, we recall two results from [32].

Proposition 5.5 ([32, Prop. 5.2.1]). *Let M be a left Fréchet module of finite projective homological dimension over a Fréchet algebra A . Suppose that there exists $n \in \mathbb{N}$ such that for each projective module $P \in A\text{-mod}$ the following conditions hold:*

- (i) $\text{Ext}_A^i(M, P) = 0$ unless $i = n$;
- (ii) $\text{Ext}_A^n(M, P)$ is Hausdorff and complete;

(iii) the canonical map

$$\mathrm{Ext}_A^n(M, A) \widehat{\otimes}_A P \rightarrow \mathrm{Ext}_A^n(M, P) \quad (16)$$

is a topological isomorphism.

Put $L = \mathrm{Ext}_A^n(M, A)$. Then for each $N \in A\text{-mod}$ there exist vector space isomorphisms

$$\mathrm{Ext}_A^i(M, N) \cong \mathrm{Tor}_{n-i}^A(L, N).$$

Proposition 5.6 ([32, Theorem 5.2.4]). *Let A be a Fréchet algebra of finite bidimension. Suppose that there exists $n \in \mathbb{N}$ such that for each projective bimodule $P \in A\text{-mod-}A$ the following conditions hold:*

- (i) $\mathcal{H}^i(A, P) = 0$ unless $i = n$;
- (ii) $\mathcal{H}^n(A, P)$ is Hausdorff and complete;
- (iii) the canonical map

$$\mathcal{H}^n(A, A^e) \widehat{\otimes}_{A^e} P \rightarrow \mathcal{H}^n(A, P)$$

is a topological isomorphism.

Finally, suppose that $L = \mathcal{H}^n(A, A^e)$ is projective as a right $A\text{-}\widehat{\otimes}\text{-module}$. Then for each bimodule $M \in A\text{-mod-}A$ there exist vector space isomorphisms

$$\mathcal{H}^i(A, M) \cong \mathcal{H}_{n-i}^n(A, L \widehat{\otimes}_A M).$$

We can now simplify conditions (i)–(iii) of Propositions 5.5 and 5.6 as follows.

Corollary 5.7. *Let M be a left Fréchet module of strictly finite type and of finite projective homological dimension over a nuclear Fréchet algebra A . Suppose that there exists $n \in \mathbb{N}$ such that $\mathrm{Ext}_A^i(M, A) = 0$ unless $i = n$, and that $L = \mathrm{Ext}_A^n(M, A)$ is Hausdorff. Then for each $N \in A\text{-mod}$ there exist vector space isomorphisms*

$$\mathrm{Ext}_A^i(M, N) \cong \mathrm{Tor}_{n-i}^A(L, N).$$

Proof. Lemma 5.4 shows that if conditions (i) and (ii) of Proposition 5.5 hold for $P = A$, then conditions (i)–(iii) hold for each flat $P \in A\text{-mod}$. The rest is clear. \square

The following result is an analytic version of Van den Bergh's theorem [39]. It follows from Proposition 5.6 in exactly the same way as Corollary 5.7 follows from Proposition 5.5.

Corollary 5.8. *Let A be a nuclear Fréchet algebra of finite type and of finite bidimension. Suppose that there exists $n \in \mathbb{N}$ such that $\mathcal{H}^i(A, A^e) = 0$ unless $i = n$, and that $L = \mathcal{H}^n(A, A^e)$ is Hausdorff and invertible as a Fréchet A -bimodule. Then A satisfies $\mathrm{VdB}(n)$ with dualizing bimodule L .*

Our next goal is to apply the above results to the algebra $\mathcal{O}(X)$.

Lemma 5.9. *Let X be a Stein manifold, and let \mathcal{F} and \mathcal{G} be coherent \mathcal{O}_X -modules. If \mathcal{F} is locally free, then the canonical isomorphism*

$$\mathbf{h}_{\mathcal{O}(X)}(\mathcal{F}(X), \mathcal{G}(X)) \cong \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))$$

is a topological isomorphism.

Proof. The case where $\mathcal{F} \cong \mathcal{O}_X$ is clear. In the general case \mathcal{F} is isomorphic to a direct summand of \mathcal{O}_X^p for some $p \in \mathbb{N}$ (see [11, 6.2 and 6.3]), so the result follows by additivity and functoriality. \square

Let X be a complex manifold, and let $Y \subset X$ be a closed submanifold. Denote by \mathcal{I} the ideal sheaf of Y in X . Recall that the *normal sheaf* of Y in X is defined to be $\mathcal{N}_{Y|X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$.

Proposition 5.10. *Let X be a Stein manifold, and let $Y \subset X$ be a closed submanifold of codimension m . Suppose that both X and Y are of Liouville type. Then for each flat Fréchet $\mathcal{O}(X)$ -module F and each $i \in \mathbb{Z}_+$ there exist topological isomorphisms*

$$\mathrm{Ext}_{\mathcal{O}(X)}^i(\mathcal{O}(Y), F) \cong \begin{cases} \Gamma(X, \bigwedge^m \mathcal{N}_{Y|X}) \widehat{\otimes}_{\mathcal{O}(X)} F & \text{for } i = m, \\ 0 & \text{otherwise.} \end{cases} \quad (17)$$

Proof. Let $\mathcal{P} \rightarrow \mathcal{O}_Y \rightarrow 0$ be a Forster resolution of \mathcal{O}_Y over \mathcal{O}_X . By [16, III.7.2], we have

$$\mathcal{E}xt_{\mathcal{O}_X}^i(\mathcal{O}_Y, \mathcal{O}_X) \cong \begin{cases} \bigwedge^m \mathcal{N}_{Y|X} & \text{for } i = m, \\ 0 & \text{otherwise} \end{cases}$$

(the proof of this fact, given in [16] for schemes, applies to complex manifolds without changes). Therefore we have an exact sequence

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{P}, \mathcal{O}_X) \rightarrow \bigwedge^m \mathcal{N}_{Y|X} \rightarrow 0.$$

Applying the section functor and taking into account Lemma 5.9, we obtain an exact sequence

$$\mathbf{h}_{\mathcal{O}(X)}(\mathcal{P}(X), \mathcal{O}(X)) \rightarrow \Gamma(X, \bigwedge^m \mathcal{N}_{Y|X}) \rightarrow 0$$

of Fréchet $\mathcal{O}(X)$ -modules. This yields isomorphisms (17) for $F = \mathcal{O}(X)$. The general case follows from Lemma 5.4. \square

Let X be a complex manifold embedded into $X \times X$ via the diagonal embedding. Recall that the normal sheaf $\mathcal{N}_{X|X \times X}$ is then isomorphic to the tangent sheaf \mathcal{T}_X of X (cf. the proof of Corollary 3.9). For each $n \in \mathbb{N}$, let $T^n(X) = \Gamma(X, \bigwedge^n \mathcal{T}_X)$ denote the space of holomorphic n -polyvector fields on X . Now Proposition 5.10 implies the following.

Proposition 5.11. *Let X be an n -dimensional Stein manifold of Liouville type. Then for each flat Fréchet $\mathcal{O}(X)$ -bimodule F and each $i \in \mathbb{Z}_+$ there exist topological isomorphisms*

$$\mathcal{H}^i(\mathcal{O}(X), F) \cong \begin{cases} T^n(X) \widehat{\otimes}_{\mathcal{O}(X)^e} F & \text{for } i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 5.12. *Let X be a Stein manifold, and let $Y \subset X$ be a closed submanifold of codimension m . Suppose that both X and Y are of Liouville type. Then for each Fréchet $\mathcal{O}(X)$ -module M there exist vector space isomorphisms*

$$\mathrm{Ext}_{\mathcal{O}(X)}^i(\mathcal{O}(Y), M) \cong \mathrm{Tor}_{m-i}^{\mathcal{O}(X)}(\Gamma(X, \bigwedge^m \mathcal{N}_{Y|X}), M). \quad (18)$$

Proof. Apply Proposition 5.10 and Corollary 5.7. \square

Theorem 5.13. *Let X be an n -dimensional Stein manifold of Liouville type. Then $\mathcal{O}(X)$ satisfies VdB(n) with dualizing bimodule $T^n(X)$.*

Proof. Apply Proposition 5.11 and Corollary 5.8. \square

Remark 5.14. In [32, 5.4.1], we proved (18) in the situation where X is a smooth real manifold, $\mathcal{O}(X) = C^\infty(X)$ is the Fréchet algebra of smooth functions on X , and $Y \subset X$ is a closed submanifold of codimension m . This clearly implies that $\mathrm{dh}_{C^\infty(X)} C^\infty(Y) = m$ [32, 5.4.2]. The fact that $\mathrm{w.dh}_{C^\infty(X)} C^\infty(Y) = m$ was not stated explicitly in [32], but it also easily follows from the above-mentioned “smooth” version of (18). Indeed, letting $A = C^\infty(X)$, $B = C^\infty(Y)$, and $L = \Gamma(X, \bigwedge^m \mathcal{K}_{Y|X})$, we obtain

$$\mathrm{Tor}_m^A(L, B) \cong \mathrm{Ext}_A^0(B, B) = \mathbf{h}_A(B, B) = \mathbf{h}_B(B, B) \cong B \neq 0,$$

whence $\mathrm{w.dh}_A B \geq m$. Since $\mathrm{dh}_A B = m$ (see above), we conclude that $\mathrm{w.dh}_A B = m$ as well.

6. HYPERCONVEX SUBMANIFOLDS AND PROJECTIVE DIMENSION

Let X be a complex manifold. Recall from [37] that X is *hyperconvex* if there exists a plurisubharmonic function $\rho: X \rightarrow [-\infty, 0)$ such that for each $c < 0$ the set $\overline{\{x \in X : \rho(x) \leq c\}}$ is compact. For example, each analytic polyhedron in a Stein manifold [37] and each relatively compact pseudoconvex domain with a C^2 boundary in a Stein manifold [6] are hyperconvex. More examples can be found in [3]. Note that a hyperconvex manifold X can never be of Liouville type (unless $\dim X = 0$).

By definition [43] (see also [25]), a Fréchet space E has property $(\overline{\Omega})$ if the topology on E can be determined by an increasing sequence $\{\|\cdot\|_n : n \in \mathbb{N}\}$ of seminorms satisfying the following condition: for each $p \in \mathbb{N}$ there exists $q \in \mathbb{N}$ such that for each $k \in \mathbb{N}$ there exists $C > 0$ satisfying

$$\|y\|_q^* \leq C(\|y\|_p^*)^{1/2}(\|y\|_k^*)^{1/2} \quad (y \in E^*).$$

Clearly, $(\overline{\Omega})$ implies (Ω) (see Section 4), but not vice versa. If X is a Stein manifold, then $\mathcal{O}(X)$ has $(\overline{\Omega})$ if and only if X is hyperconvex [2, 47]. For more results in this area, we refer to [3].

Let E and F be locally convex spaces. Following [44], we say that a linear map $\varphi: E \rightarrow F$ is *bounded* if there exists a 0-neighborhood $U \subset E$ such that $\varphi(U) \subset F$ is bounded. Clearly, each bounded linear map is continuous. On the other hand, the identity map on a locally convex space E is bounded if and only if E is normable. In this section, we will use the following important result by D. Vogt [44] (see also [25, 29.21]): if E is a Fréchet space with $(\overline{\Omega})$ and F is a Fréchet space with (DN) , then each continuous linear map from E to F is bounded.

The set of all bounded linear maps from E to F is denoted by $\mathcal{LB}(E, F)$; obviously, this is a vector subspace of $\mathcal{L}(E, F)$. Note also that $\mathcal{LB}(E, F)$, like $\mathcal{L}(E, F)$, is a bifunctor on the category of locally convex spaces with values in the category of vector spaces.

Recall that an inverse system $(E_\lambda, \tau_\mu^\lambda)$ of locally convex spaces is *reduced* if the canonical projections $\tau_\mu: \varprojlim E_\lambda \rightarrow E_\mu$ have dense ranges. A *reduced inverse limit* is the inverse limit of a reduced inverse system of locally convex spaces.

Lemma 6.1. *Let $E = \varprojlim E_\lambda$ be a reduced inverse limit of locally convex spaces. Then for each Hausdorff l.c.s. F the canonical maps $\mathcal{LB}(E_\lambda, F) \rightarrow \mathcal{LB}(E, F)$ yield a vector space isomorphism*

$$\varinjlim \mathcal{LB}(E_\lambda, F) \rightarrow \mathcal{LB}(E, F). \quad (19)$$

Proof. Since the projections $\tau_\lambda: E \rightarrow E_\lambda$ have dense ranges, it follows that the maps $\mathcal{LB}(E_\lambda, F) \rightarrow \mathcal{LB}(E, F)$ are injective, and hence (19) is injective as well. Let now $\varphi \in \mathcal{LB}(E, F)$, and let $U \subset E$ be a 0-neighborhood such that $B = \varphi(U)$ is bounded in F . Without loss of generality we may assume that $U = \tau_\lambda^{-1}(V)$ for some λ and for some open 0-neighborhood $V \subset E_\lambda$. Set $E_\lambda^0 = \text{Im } \tau_\lambda \subset E_\lambda$, and define $\varphi_\lambda^0: E_\lambda^0 \rightarrow F$ by

$$\varphi_\lambda^0(\tau_\lambda(x)) = \varphi(x) \quad (x \in E).$$

To see that φ_λ^0 is well defined, assume that $\tau_\lambda(x) = 0$. Then for each $\varepsilon > 0$ we have $\tau_\lambda(x) \in \varepsilon V$, whence $x \in \varepsilon U$ and $\varphi(x) \in \varepsilon B$. Since F is Hausdorff, this implies that $\varphi(x) = 0$, showing that φ_λ^0 is well defined. Now observe that

$$\varphi_\lambda^0(V \cap E_\lambda^0) = \varphi_\lambda^0(\tau_\lambda(U)) = B,$$

and so φ_λ^0 is bounded. Since E_λ^0 is dense in E_λ , φ_λ^0 uniquely extends to a continuous linear map $\varphi_\lambda: E_\lambda \rightarrow F$. Moreover, since $V \cap E_\lambda^0$ is dense in V , it follows that $\varphi_\lambda(V) \subset \overline{B}$, and so φ_λ is bounded. It remains to observe that φ is the image of φ_λ under the canonical map $\mathcal{LB}(E_\lambda, F) \rightarrow \mathcal{LB}(E, F)$. Therefore (19) is onto. \square

Before formulating the next result, recall from [14, II.3.1] that, if E is a Banach space and F is a nuclear Fréchet space, then there exists a topological isomorphism

$$F \widehat{\otimes} E^* \rightarrow \mathcal{L}(E, F), \quad y \otimes f \mapsto (x \mapsto f(x)y). \quad (20)$$

If, in addition, E and F are left Fréchet A -modules, then it is easy to check that (20) is an A -bimodule isomorphism.

Lemma 6.2. *Let A be a Fréchet-Arens-Michael algebra of finite type. Suppose that A satisfies $\text{VdB}(n)$ with nuclear dualizing bimodule L . Let now $M \in A\text{-mod}$, and assume that all continuous linear maps from M to L^{-1} are bounded. Then there exists a vector space isomorphism*

$$\text{Ext}_A^n(M, L^{-1}) \cong M^*. \quad (21)$$

As a consequence, if $M \neq 0$, then $\text{dh}_A M = n$.

Proof. First suppose that M is a Banach A -module. Using (3), (4), property $\text{VdB}(n)$, and (20), we obtain the following vector space isomorphisms:

$$\begin{aligned} \text{Ext}_A^n(M, L^{-1}) &\cong \mathcal{H}^n(A, \mathcal{L}(M, L^{-1})) \cong \mathcal{H}_0(A, L \widehat{\otimes}_A \mathcal{L}(M, L^{-1})) \\ &\cong \mathcal{H}_0(A, L \widehat{\otimes}_A L^{-1} \widehat{\otimes}_A M^*) \cong \mathcal{H}_0(A, A \widehat{\otimes}_A M^*) \cong \text{Tor}_0^A(M^*, A) \cong M^*. \end{aligned}$$

This proves (21) in the case where M is a Banach module.

In the general case, represent M as a reduced inverse limit $M = \varprojlim M_\lambda$ of left Banach A -modules (see [34, Prop. 3.5]). By Lemma 6.1, we have vector space isomorphisms

$$\mathcal{L}(M, L^{-1}) = \mathcal{LB}(M, L^{-1}) \cong \varprojlim \mathcal{LB}(M_\lambda, L^{-1}) = \varprojlim \mathcal{L}(M_\lambda, L^{-1}).$$

In other words, for $P = A \widehat{\otimes} A$ we have a vector space isomorphism

$$\mathbf{h}_A(P \widehat{\otimes}_A M, L^{-1}) \cong \varprojlim \mathbf{h}_A(P \widehat{\otimes}_A M_\lambda, L^{-1}). \quad (22)$$

By additivity and functoriality, (22) holds for each finitely generated, strictly projective Fréchet A -bimodule P .

Now let $P \rightarrow A \rightarrow 0$ be a resolution of A in $A\text{-mod-}A$ such that all the P_i 's are finitely generated and strictly projective. Then for each $N \in A\text{-mod}$ $P \widehat{\otimes}_A N \rightarrow N \rightarrow 0$ is a

projective resolution of N in $A\text{-mod}$. Using (22) and the already established isomorphism (21) for the Banach modules M_λ , we obtain vector space isomorphisms

$$\begin{aligned} \text{Ext}_A^n(M, L^{-1}) &= H^n(\mathbf{h}_A(P \hat{\otimes}_A M, L^{-1})) \cong \varinjlim H^n(\mathbf{h}_A(P \hat{\otimes}_A M_\lambda, L^{-1})) \\ &= \varinjlim \text{Ext}_A^n(M_\lambda, L^{-1}) \cong \varinjlim M_\lambda^* \cong M^*. \end{aligned}$$

This proves (21), which in turn implies that $\text{dh}_A M \geq n$ provided that $M \neq 0$. On the other hand, $\text{VdB}(n)$ readily implies that¹ $\text{db } A \leq n$, whence $\text{dh}_A M \leq n$. This completes the proof. \square

Theorem 6.3. *Let X be a Stein manifold, and let Y be a closed submanifold of X . Suppose that X is of Liouville type and that Y is hyperconvex. Then $\text{dh}_{\mathcal{O}(X)} \mathcal{O}(Y) = \dim X$.*

Proof. Since X is of Liouville type, it follows that $\mathcal{O}(X)$ has property (DN) [2, 45, 47]. Moreover, if \mathcal{F} is a locally free \mathcal{O}_X -module, then \mathcal{F} is a direct summand of \mathcal{O}_X^p for some p [11, 6.2 and 6.3], and so $\mathcal{F}(X)$ has property (DN) . In particular, the space $\Omega^n(X)$ of holomorphic n -forms has property (DN) , where $n = \dim X$. On the other hand, since Y is hyperconvex, it follows that $\mathcal{O}(Y)$ has property $(\overline{\Omega})$ [2, 47], and so all continuous linear maps from $\mathcal{O}(Y)$ to $\Omega^n(X)$ are bounded [44]. Taking into account Theorem 5.13, we see that the algebra $A = \mathcal{O}(X)$ and the modules $M = \mathcal{O}(Y)$, $L^{-1} = \Omega^n(X)$ satisfy the conditions of Lemma 6.2. Therefore $\text{dh}_{\mathcal{O}(X)} \mathcal{O}(Y) = n$, as required. \square

Remark 6.4. By comparing Corollary 3.2 and Theorem 6.3, we see that, under the conditions of Theorem 6.3, there is no *admissible* Forster resolution of $\mathcal{O}(Y)$ over $\mathcal{O}(X)$ (unless $\dim Y = 0$). This can also be seen directly as follows. If resolution (8) were admissible, then the underlying Fréchet space of $\mathcal{O}(Y)$ would be a direct summand of P_0 and would have $(\overline{\Omega})$ and (DN) simultaneously, which is impossible for a nonnormable space.

Example 6.5. Let Y be a nonsingular affine algebraic variety in \mathbb{C}^n . Using Theorems 3.10 and 4.3, we see that

$$\text{dh}_{\mathcal{O}(\mathbb{C}^n)} \mathcal{O}(Y) = \text{w.dh}_{\mathcal{O}(\mathbb{C}^n)} \mathcal{O}(Y) = n - \dim Y.$$

Let now Y be any hyperconvex Stein manifold (for example, a polydisk or a ball in \mathbb{C}^m). By the Remmert-Bishop-Narasimhan Embedding Theorem, Y can be embedded into \mathbb{C}^n for $n = 2 \dim Y + 1$. Using Theorems 3.10 and 6.3, we see that

$$\text{w.dh}_{\mathcal{O}(\mathbb{C}^n)} \mathcal{O}(Y) = n - \dim Y, \quad \text{but } \text{dh}_{\mathcal{O}(\mathbb{C}^n)} \mathcal{O}(Y) = n.$$

In conclusion, let us formulate an open problem due to A. Ya. Helemskii [20]:

Problem 6.6 (cf. Corollaries 3.11 and 4.4). *Let X be a Stein manifold. Is it true that $\text{dg } \mathcal{O}(X) = \dim X$ or $\text{db } \mathcal{O}(X) = \dim X$?*

Acknowledgements. The author thanks A. Ya. Helemskii and P. Domański for helpful discussions.

¹In fact, it is easy to show that if A satisfies $\text{VdB}(n)$, then $\text{db } A = \text{w.db } A = n$ [35]. If, in addition, A is nuclear and locally m -convex, then $\text{dg } A = \text{w.dg } A = n$ as well [loc. cit.].

REFERENCES

- [1] H. Alexander, *Explicit imbedding of the (punctured) disc into \mathbb{C}^2* , Comment. Math. Helv. **52** (1977), no. 4, 539–544.
- [2] A. Aytuna, *On the linear topological structure of spaces of analytic functions*, Proceedings of the functional analysis conference (Silivri/Istanbul, 1985), Doğa Mat. **10** (1986), no. 1, Special Issue, 46–49.
- [3] A. Aytuna, *Stein spaces M for which $\mathcal{O}(M)$ is isomorphic to a power series space*, Advances in the theory of Fréchet spaces (Istanbul, 1988), 115–154, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 287, Kluwer Acad. Publ., Dordrecht, 1989.
- [4] A. Aytuna, A. Sadullaev, *Parabolic Stein manifolds*, Preprint arXiv:1112.1626v1 [math.CV].
- [5] N. Ciccoli, *The modular class and its quantization – minicourse*, Univ. Iagel. Acta Math. No. 47 (2009), 29–50.
- [6] K. Diederich and J. E. Fornaess, *Pseudoconvex domains: bounded strictly plurisubharmonic exhaustion functions*, Invent. Math. **39** (1977), no. 2, 129–141.
- [7] V. Dolgushev, *The Van den Bergh duality and the modular symmetry of a Poisson variety*, Selecta Math. (N.S.) **14** (2009), no. 2, 199–228.
- [8] V. Dolgushev, D. Tamarkin, and B. Tsygan, *Formality theorems for Hochschild complexes and their applications*, Lett. Math. Phys. **90** (2009), no. 1–3, 103–136.
- [9] J. Eschmeier and M. Putinar, *Spectral decompositions and analytic sheaves*, London Mathematical Society Monographs, New Series, 10, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1996.
- [10] M. Farinati, *Hochschild duality, localization, and smash products*, J. Algebra **284** (2005), no. 1, 415–434.
- [11] O. Forster, *Zur Theorie der Steinschen Algebren und Moduln*, Math. Z. **97** (1967), 376–405.
- [12] V. Ginzburg, *Calabi-Yau algebras*, Preprint arXiv:math/0612139v3.
- [13] H. Grauert and R. Remmert, *Theory of Stein spaces*, Springer-Verlag, Berlin-New York, 1979.
- [14] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. 1955, no. 16, 140 pp.
- [15] A. Grothendieck, *Techniques de construction en géométrie analytique, VII. Étude locale des morphismes: éléments de calcul infinitésimal*, Séminaire Henri Cartan, tome 13, no. 2 (1960/1961), exp. 14, 1–27.
- [16] R. Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64, with an appendix by P. Deligne, Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin-New York, 1966.
- [17] R. Hartshorne, *Algebraic Geometry*, Graduate Texts in Mathematics, No. 52, Springer-Verlag, New York–Heidelberg, 1977.
- [18] A. Ya. Helemskii, *The Homology of Banach and Topological Algebras*, Moscow University Press, 1986 (in Russian); English transl.: Kluwer Academic Publishers, Dordrecht, 1989.
- [19] A. Ya. Helemskii, *Banach and Polynormed Algebras: General Theory, Representations, Homology*, Nauka, Moscow, 1989 (in Russian); English transl.: Oxford University Press, 1993.
- [20] A. Ya. Helemskii, *31 problems of the homology of the algebras of analysis*, “Linear and Complex Analysis: Problem Book 3, Part I” (eds. V. P. Havin and N. K. Nikolski), Lecture Notes in Math. **1573**, 54–78, Berlin, Springer, 1994.
- [21] A. Ya. Helemskii, *Homology for the Algebras of Analysis*, in: Handbook of algebra, Vol. 2 (ed. M. Hazewinkel), 151–274, Amsterdam, North-Holland, 2000.
- [22] G. Hochschild, B. Kostant, and A. Rosenberg, *Differential forms on regular affine algebras*, Trans. Amer. Math. Soc. **102** (1962), 383–408.
- [23] T. Lambre, *Dualité de Van den Bergh et structure de Batalin-Vilkoviskii sur les algèbres de Calabi-Yau*, J. Noncommut. Geom. **4** (2010), no. 3, 441–457.
- [24] Yu. I. Manin, *Lectures in Algebraic Geometry. Part II: The K -functor in Algebraic Geometry* (Russian), Moscow Univ. Press, Moscow, 1971.
- [25] R. Meise and D. Vogt, *Introduction to Functional Analysis*, Oxford Graduate Texts in Mathematics, 2, The Clarendon Press, Oxford University Press, New York, 1997.

- [26] L. Menichi, *Van den Bergh isomorphisms in string topology*, J. Noncommut. Geom. **5** (2011), no. 1, 69–105.
- [27] B. S. Mitjagin and G. M. Henkin, *Linear problems of complex analysis* (Russian), Uspehi Mat. Nauk **26** (1971), no. 4 (160), 93–152; English transl.: Russ. Math. Surv., 1971, 26 (4), 99–164.
- [28] D. G. Northcott, *An introduction to homological algebra*, Cambridge University Press, New York, 1960.
- [29] D. G. Northcott, *A first course of homological algebra*, Cambridge University Press, London, 1973.
- [30] V. P. Palamodov, *On the Liouville property of section spaces of coherent analytic sheaves*, Functional analysis (Essen, 1991), 459–473, Lecture Notes in Pure and Appl. Math., 150, Dekker, New York, 1994.
- [31] A. Yu. Pirkovskii, *On certain homological properties of Stein algebras*, Functional analysis, 3, J. Math. Sci. (New York) **95** (1999), no. 6, 2690–2702.
- [32] A. Yu. Pirkovskii, *Injective topological modules, additivity formulas for homological dimensions, and related topics*, Topological Homology: Helemskii's Moscow Seminar (ed. A. Ya. Helemskii), 93–143, Nova Science Publishers, Inc., Huntington, NY, 2000.
- [33] A. Yu. Pirkovskii, *Weak homological dimensions and biflat Köthe algebras*, Mat. Sb. **199** (2008), No. 5, 45–80 (Russian); English transl.: Sbornik: Mathematics **199** (2008), No. 5, 673–705.
- [34] A. Yu. Pirkovskii, *Arens-Michael envelopes, homological epimorphisms, and relatively quasi-free algebras* (Russian), Tr. Mosk. Mat. Obs. **69** (2008), 34–125; English transl.: Trans. Moscow Math. Soc. 2008, 27–104.
- [35] A. Yu. Pirkovskii, *Homological dimensions and Van den Bergh isomorphisms for nuclear Fréchet algebras* (Russian), Izvestiya: Mathematics, to appear (2012).
- [36] M. Putinar, *Functional calculus with sections of an analytic space*, J. Operator Theory **4** (1980), no. 2, 297–306.
- [37] J.-L. Stehlé, *Fonctions plurisousharmoniques et convexité holomorphe de certains fibrés analytiques*, Séminaire Pierre Lelong (Analyse), Année 1973–1974, 155–179, Lecture Notes in Math., 474, Springer, Berlin, 1975.
- [38] J. L. Taylor, *Homology and cohomology for topological algebras*, Adv. Math. **9** (1972), 137–182.
- [39] M. Van den Bergh, *A relation between Hochschild homology and cohomology for Gorenstein rings*, Proc. Amer. Math. Soc. **126** (1998), no. 5, 1345–1348; erratum: Proc. Amer. Math. Soc. **130** (2002), no. 9, 2809–2810.
- [40] D. Vogt, *Charakterisierung der Unterräume von s* , Math. Z. **155** (1977), no. 2, 109–117.
- [41] D. Vogt and M. J. Wagner, *Charakterisierung der Quotientenräume von s und eine Vermutung von Martineau*, Studia Math. **67** (1980), no. 3, 225–240.
- [42] D. Vogt and M. J. Wagner, *Charakterisierung der Unterräume und Quotientenräume der nuklearen stabilen Potenzreihenräume von unendlichem Typ*, Studia Math. **70** (1981), no. 1, 63–80.
- [43] D. Vogt, *Eine Charakterisierung der Potenzreihenräume von endlichem Typ und ihre Folgerungen*, Manuscripta Math. **37** (1982), no. 3, 269–301.
- [44] D. Vogt, *Frécheträume, zwischen denen jede stetige lineare Abbildung beschränkt ist*, J. Reine Angew. Math. **345** (1983), 182–200.
- [45] D. Vogt, *Some results on continuous linear maps between Fréchet spaces*, Functional analysis: surveys and recent results, III (Paderborn, 1983), 349–381, North-Holland Math. Stud., 90, North-Holland, Amsterdam, 1984.
- [46] V. P. Zakharyuta, *Spaces of analytic functions on algebraic varieties in \mathbb{C}^n* (Russian), Izv. Severo-Kavkaz. Naučn. Centra Vysš. Školy Ser. Estestv. Nauk. 1977, no. 4, 52–55, 117.
- [47] V. P. Zakharyuta, *Isomorphism of spaces of analytic functions* (Russian), Dokl. Akad. Nauk SSSR **255** (1980), no. 1, 11–14.

FACULTY OF MATHEMATICS, NATIONAL RESEARCH UNIVERSITY HIGHER SCHOOL OF ECONOMICS,
VAVILOVA 7, 117312 MOSCOW, RUSSIA

E-mail address: aupirkovskii@hse.ru, pirkosha@online.ru